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EXTENDED DOUBLE-COMPLEX LINEAR SYSTEMS AND NEW DOUBLE INFINITE-DIMENSIONAL HIDDEN SYMMETRIES FOR THE EINSTEIN–KALB–RAMOND THEORY

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By using a so-called extended double (ED)-complex method, the previously found doubleness symmetry of the dimensionally reduced Einstein–Kalb–Ramond (EKR) theory is further exploited. A $2d \times 2d$ matrix double-complex H-potential is constructed and the field equations are written in a double-complex formulation. A pair of ED-complex Hauser–Ernst-type linear systems are established. Based on these linear systems, explicit formulations of new double hidden symmetry transformations for the EKR theory are given. These symmetry transformations are verified to constitute double infinite-dimensional Lie algebras, each of which is a semidirect product of the Kac–Moody $\widehat{o(d,d)}$ and Virasoro algebras (without center charges). These results demonstrate that the EKR theory under consideration possesses richer symmetry structures than previously expected, and the ED-complex method is necessary and more effective.

Keywords: Einstein–Kalb–Ramond theory; extended double-complex method; infinite-dimensional double symmetry algebra.

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1. Introduction

Owing to their importance in theoretical and mathematical physics, the studies of symmetries for the dimensionally reduced low energy effective (super)string theories have attracted much attention in the recent past (see e.g. Refs. 1–26). Such effective string theories describe various interacting matter fields coupled to gravity, the effective heterotic string theory describing the Einstein and Kalb–Ramond (EKR) fields[15,16] is a typical model of this kind. Some symmetries for the EKR theory have been found and some analogies between it and the reduced vacuum Einstein theory have been noted. However, many *scalar* functions in pure gravity correspond, formally, to *matrix* ones in the EKR theory, thus the noncommuting property of the matrices gives rise to essential complications for the further study of the latter.

Moreover, some particular relations, such as for any 2×2 matrix A: $A^\top \epsilon A = (\det A)\epsilon$, $A^\top \epsilon + \epsilon A = (\operatorname{tr} A)\epsilon$ $\left(\text{with } \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ have no general analogues for higher-dimensional $n \times n$ ($n \geq 3$) matrices, while these relations are useful and important in some studies of the reduced vacuum gravity (e.g. Refs. 27–31). Since in the studies of the EKR theory, we deal mainly with $2d \times 2d$ function matrices and in general $d \geq 2$, so some deeper researches and further extended studying methods are needed.

The present paper is a continuation of our previous paper.[26] In this paper, by using a so-called extended double (ED)-complex function method,[32] the previously found doubleness symmetry[26] of the EKR theory is further exploited. A double-complex $2d \times 2d$ matrix H-potential is constructed and the motion equations are extended into a double-complex form in terms of this H-potential. Moreover, we further find that a pair of ED-complex Hauser–Ernst (HE)-type linear systems can be established, and based on these linear systems, new infinitesimal double symmetry transformations for the EKR theory are explicitly constructed. Then these symmetry transformations are verified to constitute double infinite-dimensional Lie algebras, each of which is a semidirect product of the Kac–Moody $\widehat{o(d,d)}$ and Virasoro algebras (without center charges). These results demonstrate that the theory under consideration possesses richer symmetry structures than previously expected, and the ED-complex method is necessary and more effective.

In Sec. 2, some related concepts and notations of the ED-complex functions[32] and the double-complex matrix Ernst formulation of the EKR field equations[26] are briefly recalled. In Sec. 3, a double-complex matrix H-potential is constructed and a pair of ED-complex HE-type linear systems are established. In Sec. 4, by virtue of these linear systems, we give explicit expressions of some infinitesimal double transformations for the studied theory and then verify that these transformations are all double hidden symmetries of the EKR theory. The double infinite-dimensional Lie algebra structure of these hidden symmetries is calculated out in Sec. 5. Finally, Sec. 6 gives some summary and discussions.

2. ED-Complex Function and Double-Complex Matrix Ernst Equations of the EKR Theory

For the later use, here we briefly recall some related concepts and notations of the ED-complex function[32] and the double-complex matrix Ernst formulation of the EKR field equations.[26]

2.1. *ED-complex function*[32]

Let i and J denote, respectively, the ordinary and the ED imaginary unit. We shall concern ourselves mainly with some special values of J, i.e. $J = j$ ($j^2 = -1$, $j \neq \pm i$) or $J = \varepsilon$ ($\varepsilon^2 = +1$, $\varepsilon \neq \pm 1$). If a series $\sum_{n=0}^{\infty} |a_n|$, $a_n \in \mathbf{C}$ (ordinary complex number) is convergent, then $a(J) = \sum_{n=0}^{\infty} a_n J^{2n}$ is called an ED ordinary complex number, which can correspond to a pair (a_C, a_H) of ordinary complex number, where $a_C := a(J = j)$, $a_H := a(J = \varepsilon)$. When $a(J)$ and $b(J)$ both are ED ordinary complex numbers,

$$c(J) = a(J) + Jb(J) \tag{2.1}$$

is called an ED-complex number, it can correspond to a pair (c_C, c_H), where $c_C := c(J = j) = a_C + jb_C$, $c_H := c(J = \varepsilon) = a_H + \varepsilon b_H$. If $a(J)$ and $b(J)$ are real, we call them double-real and call the corresponding $c(J)$ simply a double-complex number.[33]

We would like to point out that, from the above definitions, J should be taken as an indeterminate rather than a discrete variable. The ED-complex method can be regarded as some "deformation" theory, in which J plays the role of "deformation parameter" (or analytical link, cf. Ref. 33 for nonextended case). By doubleness symmetry we in fact mean the symmetry property of the considered theory under this "deformation." We call it an ED-complex method only because in most of its applications (e.g. in the present paper) we are mainly interested in the cases of $J = j$ and $J = \varepsilon$.

All ED-complex numbers with usual addition and multiplication constitute a commutative ring. Corresponding to the two imaginary units J and i in this ring, we have two complex conjugations: ED-complex conjugation "$\star$" and ordinary complex conjugation "$-$":

$$c(J)^\star := a(J) - Jb(J)\,, \qquad \overline{c(J)} := \overline{a(J)} + J\overline{b(J)}\,. \tag{2.2}$$

These imply that $J^\star = -J$, $\bar{J} = J$, $i^\star = i$, $\bar{i} = -i$. If $a(J)$ and $b(J)$ are ED ordinary complex functions of some ordinary complex variables $z_1, \ldots, z_n$, then $c(z_1, \ldots, z_n; J) = a(z_1, \ldots, z_n; J) + Jb(z_1, \ldots, z_n; J)$ is called an ED-complex function. We say $c(z_1, \ldots, z_n; J)$ to be continuous, analytical, etc. if $a(z_1, \ldots, z_n; J)$ and $b(z_1, \ldots, z_n; J)$ both, as ordinary complex functions, have the same properties. We also need ED-complex (function) matrices, and for an ED-complex matrix $W(J)$, we define

$$W(J)^+ := [W(J)^\star]^\top\,, \tag{2.3}$$

"$\top$" denotes the transposition.

2.2. *Double-complex matrix Ernst formulation of the EKR field equations*[26]

We start with the action describing a system arising in the low energy limit of heterotic string theory as

$$\mathcal{S} = \int \left[\mathcal{R} + \mathcal{G}^{MN}\partial_M\Phi\partial_N\Phi - \frac{1}{12}\,\mathcal{H}_{MNL}\mathcal{H}^{MNL}\right] e^{-\Phi}\sqrt{|\mathcal{G}|}d^{3+d}x\,, \tag{2.4}$$

where $\mathcal{R}$ is the Ricci scalar for the metric $\mathcal{G}_{MN}$ $(M, N = 0, 1, \ldots, 2+d)$, Φ is the dilaton field and

$$\mathcal{H}_{MNL} = \partial_M \mathcal{B}_{NL} + \text{cyclic}\,, \tag{2.5}$$

while $\mathcal{B}_{NL}$ is antisymmetric Kalb–Ramond field and $\mathcal{H}_{MNL}$ is called nondualized axion field.

According to Maharana and Schwarz[1] and Sen,[2,3] when dimensionally reducing from $3+d$ to 3 dimensions by compactification on a d-dimensional torus, one can obtain "three-dimensional fields" G_{mn}, B_{mn}, ϕ, $A_\mu^{(a)}$, $g_{\mu\nu}$ and $B_{\mu\nu}$ $(m, n = 1, 2, \ldots, d;\ \mu,\ \nu = 0, 1, 2;\ a = 1, 2, \ldots, 2d)$ through the relations

$$\begin{gathered}
G_{mn} = \mathcal{G}_{m+2,n+2}\,, \quad B_{mn} = \mathcal{B}_{m+2,n+2}\,, \quad g_{\mu\nu} = \mathcal{G}_{\mu\nu} - G^{mn}\mathcal{G}_{m+2,\mu}\mathcal{G}_{n+2,\nu}\,, \\
A_\mu^{(m)} = \frac{1}{2}G^{mn}\mathcal{G}_{n+2,\mu}\,, \quad A_\mu^{(d+m)} = \frac{1}{2}G^{mn}\mathcal{B}_{n+2,\mu} - G^{ml}B_{ln}A_\mu^{(n)}\,, \\
B_{\mu\nu} = \mathcal{B}_{\mu\nu} - 4A_\mu^{(m)}B_{mn}A_\nu^{(n)} - 2\Big(A_\mu^{(m)}G_{mn}A_\nu^{(d+n)} - A_\nu^{(m)}G_{mn}A_\mu^{(d+n)}\Big)\,, \\
\phi = \Phi - \frac{1}{2}\ln\det G\,,
\end{gathered} \tag{2.6}$$

and express the reduced theory in terms of these fields. In (2.6), G^{mn} denotes the inverse of the matrix G_{mn} and ϕ is called a shifted dilaton field. In the present paper, motivated by Ref. 15, we take the ansatz

$$\mathcal{G}_{\mu,n+2} = \mathcal{B}_{\mu,n+2} = 0\,; \quad \phi = 0\,. \tag{2.7}$$

From Eqs. (2.6), the conditions (2.7) lead further to $A_\mu^{(a)} = 0$ $(a = 1, 2, \ldots, 2d)$, and it can be seen that such restrictions do not provide any further constrains on the remainder field variables. Noted the absence of the gauge vector fields and the shifted dilaton field, we call the model under consideration as Einstein–Kalb–Ramond (EKR) theory.

Following Ref. 15, we now consider the time-independent field configurations of the above EKR theory with axisymmetry, in which the three-dimensional metric can be parametrized as

$$ds^2 = \rho^2\, dx^0\, dx^0 - e^{2\gamma}\delta_{AD}\, dx^A\, dx^D\,, \quad A, D = 1, 2\,. \tag{2.8}$$

After reducing to this case and using the fact that the antisymmetric tensor field $B_{\mu\nu}$ have no dynamics in two dimensions, in addition to the above metric fields, the set of nontrivial EKR dynamical quantities also contains $G := \{G_{mn}\}$ and $B := \{B_{mn}\}$ (both denoted by $d \times d$ matrices), and all of these fields are assumed to depend only on x^1, x^2. For simplicity, we denote x^1, x^2 by x, y in the following. In terms of these, the essential dynamical equations of the EKR theory can be written as[15]

$$d(\rho G^{-1}\,{}^*dBG^{-1}) = 0\,, \qquad d(\rho\,{}^*dG\,G^{-1} - \rho BG^{-1}\,{}^*dB\,G^{-1}) = 0\,, \tag{2.9}$$

and $\rho = \rho(x, y) > 0$ is a harmonic function in two-dimensional $\{x, y\}$. Where the notations of differential form are adopted, "$*$" is the dual operation of two-dimensional Euclidean space, and from (2.6) the real matrices G and B satisfy

$$G^\top = G\,, \qquad B^\top = -B\,. \tag{2.10}$$

Moreover, according to the Einstein equations[15,34] the field $\gamma(x, y)$ in (2.8) can be obtained by a simple integration provided G, B are known, so we shall focus our attention on Eqs. (2.9) in the following.

As pointed out in Ref. 26, the EKR theory under consideration possesses a so-called doubleness symmetry such that Eqs. (2.9) can be extended into a double-complex matrix Ernst-like formulation:

$$\rho^{-1} d(\rho\, {}^* dE(J)) = dE(J) G(J)^{-1}\, {}^* dE(J)\,, \tag{2.11}$$

where $E(J) = G(J) + JB(J)$ (with $G(J)^\top = G(J)$, $B(J)^\top = -B(J)$ both are double-real $d \times d$ matrices) is called a matrix double-complex Ernst-like potential of the EKR theory, and the wedge symbol "$\wedge$" in exterior products of differential forms is omitted for simplicity. If a solution $E(J)$ of Eq. (2.11) is known, we can obtain a pair of real solutions of the EKR theory.

3. Double-Complex *H*-Potential and ED-Complex HE-Type Linear Systems

We introduce a double-real $2d \times 2d$ matrix function $M(J) = M(x, y; J)$, which satisfies

$$M(J)^\top = M(J)\,, \tag{3.1a}$$

$$M(J)\eta M(J) = -J^2\rho^2\eta\,, \tag{3.1b}$$

$$\eta := \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}, \tag{3.1c}$$

where I_d is the d-dimensional unit matrix. Taking the decomposition of $M(J)$ as

$$M(J) = \begin{pmatrix} G(J) & -G(J)B(J) \\ B(J)G(J) & -B(J)G(J)B(J) - J^2\rho^2 G(J)^{-1} \end{pmatrix}, \tag{3.2}$$

then Eq. (2.11) can be equivalently written as

$$d(\rho^{-1} M(J)\eta\, {}^* dM(J)) = 0\,. \tag{3.3}$$

According to the spirit of Ref. 26, if a solution of Eq. (3.3) with conditions (3.1) is known, then by the decomposition (3.2), we can obtain real solutions of the EKR theory *in pairs* as follows:

$$(G, B) = (T(G_C), B_C)\,, \tag{3.4a}$$

$$(\hat{G}, \hat{B}) = (G_H, V_{G_H}(B_H))\,, \tag{3.4b}$$

where the transformations T, V are defined by

$$
\begin{aligned}
T: &\qquad G \to T(G) = \rho G^{-1}\,, \\
V: &\quad G, B \to V_G(B) = \int \rho G^{-1}(\partial_y B)G^{-1}\,dx - \rho G^{-1}(\partial_x B)G^{-1}\,dy\,,
\end{aligned}
\tag{3.5}
$$

and the existence of $V_{G_H}(B_H)$ is ensured by the $J = \epsilon$ case of Eq. (3.3). Obviously, Eqs. (3.3) and (3.1) are invariant under the global transformations: $M(J) \to P^\top M(J)P$, $P \in O(d,d)$. Of course, also as will be seen in the following, the symmetry structures of the considered EKR theory are very much richer than this.

Equation (3.3) implies that we can introduce a double-real $2d \times 2d$ matrix twist potential $Q(J) = Q(x, y; J)$ such that

$$dQ(J) = -\rho^{-1}M(J)\eta\,{}^*dM(J)\,. \tag{3.6}$$

Using (3.1), we obtain from (3.6)

$$dM(J) = -\rho^{-1}J^2M(J)\eta\,{}^*dQ(J)\,. \tag{3.7}$$

Now introducing a $2d \times 2d$ matrix double-complex H-potential

$$H(J) := M(J) + JQ(J) \tag{3.8}$$

and denoting $\Omega := J\eta$, then Eqs. (3.6) and (3.7) can be equivalently written as a single double-complex matrix equation

$$dH(J) = -\rho^{-1}M(J)\Omega\,{}^*dH(J)\,. \tag{3.9}$$

Furthermore, from (3.1) and (3.6) we have $d(Q(J) + Q(J)^\top) = 2J^2\,{}^*d\rho\,\eta$. Thus, from the harmony of $\rho(x, y)$, we can introduce another real field $z = z(x, y)$ such that ${}^*d\rho = dz$ and obtain $Q(J) + Q(J)^\top = 2J^2z\eta$. These relations and Eqs. (3.1), (3.8) imply that we can express Eq. (3.9) as

$$2(z + \rho^*)dH(J) = (H(J) + H(J)^\top)\Omega\,dH(J) \tag{3.10}$$

with (3.1), this is equivalent to (3.3). In addition, from (3.10) we can obtain

$$dH(J)^\top\Omega\,dH(J) = dH(J)^\top\Omega\,{}^*dH(J) = 0\,. \tag{3.11}$$

Now we introduce an ordinary complex parameter t and define

$$
\begin{aligned}
A(t; J) &:= I - t[H(J) + H(J)^\top]\Omega\,, \\
&\qquad (I \text{ is the } 2d\text{-dimensional unit matrix})\,,
\end{aligned}
\tag{3.12}
$$

$$\Gamma(t; J) := t\Lambda(t)^{-1}dH(J)\,, \tag{3.13}$$

$$\Lambda(t) := 1 - 2t(z + \rho^*)\,, \quad \Lambda(t)^{-1} = \lambda(t)^{-2}[1 - 2t(z - \rho^*)]\,, \tag{3.14}$$

$$\lambda(t) := [(1 - 2zt)^2 + (2\rho t)^2]^{1/2}\,, \tag{3.15}$$

then Eq. (3.10) can be rewritten as

$$tdH(J) = A(t; J)\Gamma(t; J)\,. \tag{3.16}$$

From Eqs. (3.11), (3.12) and (3.16), we can obtain $d\Gamma(t; J) = \Gamma(t; J)\Omega\Gamma(t; J)$, this is just the complete integrability condition of the following ED-complex linear differential equation:

$$dF(t; J) = \Gamma(t; J)\Omega F(t; J)\,, \tag{3.17}$$

where $F(t; J) = F(x, y, t; J)$ is a $2d \times 2d$ ED-complex matrix function of x, y and t.

Equation (3.17) does not define $F(t; J)$ uniquely, so we suppress some subsidiary conditions consistent with above equations and the requirement that $F(t; J)$ be holomorphic in a neighborhood of $t = 0$. From (3.16), (3.17) and the relation $2t\Lambda^{-1}dz = -\lambda(t)^{-1}d\lambda(t)$ we have

$$dF(0; J) = 0\,, \quad d[\dot{F}(0; J) - H(J)\Omega F(0; J)] = 0\,,$$
$$d[\lambda(t)F(t; J)^{+}\Omega F(t; J)] = 0\,, \quad d[F(t; J)^{\top}\Omega A(t; J)F(t; J)] = 0\,,$$

where $\dot{F}(t; J) := \partial F(t; J)/\partial t$ and the ED-Hermitian conjugation "+" is defined by (2.3). These equations and (3.17) determine $F(t; J)$ up to right-multiplication by an arbitrary nondegenerate $2d \times 2d$ matrix function of t, so we can use this freedom and choose the integral constants consistently such that

$$F(0; J) = I\,, \tag{3.18a}$$

$$\dot{F}(0; J) = H(J)\Omega\,, \tag{3.18b}$$

$$\lambda(t)F(t; J)^{+}\Omega F(t; J) = \Omega\,, \tag{3.19a}$$

$$F(t; J)^{\top}\Omega A(t; J)F(t; J) = \Omega\,. \tag{3.19b}$$

We call Eqs. (3.17)–(3.19) an ED-complex HE-type linear system for the EKR theory.

Besides, we can establish another ED-complex linear system of the EKR theory. Now, for another ordinary-complex parameter w, we define

$$\tilde{A}(w; J) := w - (H(J) + H(J)^{\top})\Omega\,, \tag{3.20}$$

$$\tilde{\Gamma}(w; J) := \tilde{\Lambda}(w)^{-1}dH(J)\,, \tag{3.21}$$

$$\tilde{\Lambda}(w) := w - 2(z + \rho^*)\,, \quad \tilde{\Lambda}(w)^{-1} = \tilde{\lambda}(w)^{-2}[w - 2(z - \rho^*)]\,, \tag{3.22}$$

$$\tilde{\lambda}(w) := [(w - 2z)^2 + (2\rho)^2]^{1/2}\,. \tag{3.23}$$

Then Eq. (3.10) can be rewritten as

$$dH = \tilde{A}(w; J)\tilde{\Gamma}(w; J)\,, \tag{3.24}$$

by derivations similar to the above, we have

$$d\tilde{F}(w; J) = \tilde{\Gamma}(w; J)\Omega\tilde{F}(w; J)\,, \tag{3.25}$$

and require consistently that $\tilde{F}(w;J)$ be analytic around $w=0$ and satisfy

$$\tilde{\lambda}(w)\tilde{F}(w;J)^{+}\Omega\tilde{F}(w;J)=\Omega\,, \tag{3.26a}$$

$$\tilde{F}(w;J)^{\top}\Omega\tilde{A}(w;J)\tilde{F}(w;J)=\Omega\,, \tag{3.26b}$$

where $\tilde{F}(w;J)=\tilde{F}(x,y,w;J)$ is another ED-complex $2d\times 2d$ matrix function of x, y and w.

4. Parametrized Double Symmetry Transformations

By virtue of solutions $F(t;J)$, $\tilde{F}(w;J)$ of linear systems (3.17)–(3.19) and (3.25), (3.26), we can explicitly construct parametrized double symmetry transformations for the EKR theory. At first, from definitions (3.8), (3.12)–(3.15) and (3.20)–(3.23), we may consistently choose the ED-complex matrix functions $F(t;J)$ and $\tilde{F}(w;J)$ as

$$\overline{F(t;J)}=F(\bar{t};J)\,,\qquad \overline{\tilde{F}(w;J)}=\tilde{F}(\bar{w};J) \tag{4.1}$$

(i.e. the ED-real and imaginary parts of $F(t;J)$ and $\tilde{F}(w;J)$ are double ordinary real when t and w are real) in order to ensure the reality of $M(J)$ and $Q(J)$ in the transformed $H(J)$. We shall take this choice in the following.

We consider the following infinitesimal double transformation $\delta=\delta(l)$ of potential $H(J)$:

$$\delta H(J)=-J^{2}\frac{1}{l}[F(l;J)TF(l;J)^{-1}-T]\Omega\,, \tag{4.2}$$

where l is a (finite) real parameter, $F(l;J)$ is a solution of (3.17)–(3.19) with t being replaced by l, $T=T_{a}\alpha^{a}\in o(d,d)$ (the Lie algebra of the orthogonal group $O(d,d)$), T_a are generators of $o(d,d)$, α^a are infinitesimal real constants. Thus we have relation

$$T^{\top}\Omega+\Omega T=0\,. \tag{4.3}$$

Now we prove that (4.2) is a hidden symmetry transformation of the double-complex EKR motion equation (3.10) and conditions (3.1). First, from (4.2), (4.3), (3.19a) and $T^{+}=T^{\top}$ in the real Lie algebra $o(d,d)$, we have

$$\begin{aligned}\delta H(J)-\delta H(J)^{+}&=-J^{2}\frac{1}{l}\,[F(l;J)TF(l;J)^{-1}-T]\Omega\\&\quad-J^{2}\frac{1}{l}\Omega\big[F(l;J)^{+-1}T^{\top}F(l;J)^{+}-T^{\top}\big]\\&=-J^{2}\frac{1}{l}F(l;J)[TF(l;J)^{-1}\Omega F(l;J)^{+-1}\\&\quad+F(l;J)^{-1}\Omega F(l;J)^{+-1}T^{\top}]F(l;J)^{+}\\&=-J^{2}\frac{\lambda(l)}{l}F(l;J)(T\Omega+\Omega T^{\top})F(l;J)^{+}=0\,.\end{aligned} \tag{4.4}$$

From (3.8) and the definition of $z(x,y)$, Eq. (4.4) implies that $\delta M(J)^\top = \delta M(J)$ and $\delta z = 0$.

In addition, Eqs. (3.8), (3.12) and (4.4) give $M(J) = (J^2/4l)[A(J)^\star - A(J)]\Omega$ and $\delta M(J) = (1/2)[\delta H(J) + \delta H(J)^\top]$, then from (4.2), (4.3) and (3.19b) we have

$$\begin{aligned}
&\delta M(J)\Omega M(J) + M(J)\Omega\delta M(J)\\
&\quad= \frac{J^2}{8l^2}\big[\big(J^2F(l;J)TF(l;J)^{-1} + \Omega F(l;J)^{\top-1}T^\top F(l;J)^\top\Omega\big)\big(A(J) - A(J)^\star\big)\\
&\qquad + \big(A(J) - A(J)^\star\big)\big(J^2F(l;J)TF(l;J)^{-1} + \Omega F(l;J)^{\top-1}T^\top F(l;J)^\top\Omega\big)\big]\Omega\\
&\quad= \frac{J^2}{4l^2}\big[J^2A(J)F(l;J)TF(l;J)^{-1} - J^2F(l;J)TF(l;J)^{-1}A(J)^\star\\
&\qquad - A(J)^\star\Omega F(l;J)^{\top-1}T^\top F(l;J)^\top\Omega + \Omega F(l;J)^{\top-1}T^\top F(l;J)^\top\Omega A(J)\big]\Omega\\
&\quad= \frac{J^2}{4l^2}\big[\Omega F(l;J)^{\top-1}\Omega TF(l;J)^{-1} + \Omega F(l;J)^{\top-1}T^\top\Omega F(l;J)^{-1}\\
&\qquad - \lambda(l)^2F(l;J)T\Omega F(l;J)^\top\Omega - \lambda(l)^2F(l;J)\Omega T^\top F(l;J)^\top\Omega\big]\Omega = 0\,, \qquad (4.5)
\end{aligned}$$

where the relations

$$A(J) + A(J)^\star = 2(1-2lz)\,, \qquad A(J)A(J)^\star = \lambda(l)^2 \qquad (4.6)$$

have been used. Equation (4.5) implies that, under the transformation (4.2), the condition (3.1b) is preserved and $\delta\rho = 0$.

Now we investigate the equation satisfied by $\delta H(J)$. From (4.2) and (3.17), it follows that $d(\delta H) = -(J^2/l)[\Gamma(l;J)\Omega,\, F(l;J)TF(l;J)^{-1}]\Omega$, this and (3.13), (3.10) further followed by

$$\begin{aligned}
2(z+\rho^*)d(\delta H(J)) &= (H(J) + H(J)^\top)\Omega d(\delta H(J))\\
&\quad - \frac{1}{l}[(H(J) + H(J)^\top)\Omega, F(l;J)TF(l;J)^{-1}]\Gamma(l;J)\,. \qquad (4.7)
\end{aligned}$$

On the other hand, from (4.2), (4.3), (3.12), (3.16) and (3.19b) we have

$$\begin{aligned}
&(\delta H(J) + \delta H(J)^\top)\Omega\, dH(J)\\
&\quad= -\frac{J^2}{l^2}[F(l;J)TF(l;J)^{-1}\Omega - \Omega F(l;J)^{\top-1}T^\top F(l;J)^\top]\Omega A(l;J)\Gamma(l;J)\\
&\quad= -\frac{1}{l}[(H(J) + H(J)^\top)\Omega, F(l;J)TF(l;J)^{-1}]\Gamma(l;J)\,.
\end{aligned}$$

Substituting this into Eq. (4.7), we finally obtain

$$2(z+\rho^*)d(\delta H(J)) = (H(J) + H(J)^\top)\Omega d(\delta H(J)) + (\delta H(J) + \delta H(J)^\top)\Omega\, dH(J)\,. \qquad (4.8)$$

Equations (4.8) and (4.4), (4.5) show that $H(J) + \delta H(J)$ with $\delta H(J)$ given by (4.2) satisfies the same Eq. (3.10) and conditions (3.1a), (3.1b) as $H(J)$ does, i.e. (4.2) is indeed a double symmetry transformation for the EKR theory.

Similarly, by using solution $\tilde{F}(s;J)$ of (3.25) and (3.26), we can construct another parametrized double infinitesimal symmetry transformation of the EKR theory as

$$\tilde{\delta}H(J) = J^2 s[\tilde{F}(s;J)T\tilde{F}(s;J)^{-1} - T]\Omega\,, \tag{4.9}$$

where s is a finite real parameter.

The set of symmetry transformations of the EKR theory can be further enlarged. In addition to (4.2) and (4.9), we propose two other infinitesimal double transformations

$$\Delta H(J) = J^2\sigma\dot{F}(l;J)F(l;J)^{-1}\Omega\,, \tag{4.10}$$

$$\tilde{\Delta}H(J) = -J^2\epsilon s\left[s\dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1} + \frac{1}{2}\right]\Omega\,, \tag{4.11}$$

where l, s both are finite real parameters and σ, ϵ are infinitesimal real constants.

From (4.10) and (3.19a),

$$\begin{aligned} \Delta H(J) - \Delta H(J)^{+} &= J^2\sigma[\dot{F}(l;J)F(l;J)^{-1}\Omega + \Omega F(l;J)^{+-1}\dot{F}(l;J)^{+}] \\ &= -J^2\sigma\lambda(l)^{-1}\frac{\partial}{\partial l}\lambda(l)\Omega \\ &= J^2\frac{2\sigma}{\lambda(l)^2}[z(1-2lz) - 2l\rho^2]\Omega\,, \end{aligned} \tag{4.12}$$

this and the definition of $z(x,y)$ imply $(\Delta M(J))^{\top} = \Delta M(J)$ and $\Delta z = \frac{\sigma}{\lambda(l)^2}[z(1-2lz) - 2l\rho^2]$.

Moreover, since $M(J) = \frac{1}{2}[H(J) + H(J)^{\star}]$ and $(\Delta M(J))^{\top} = \Delta M(J)$ by (4.12), we have

$$\Delta M(J) = \frac{1}{2}[\Delta H(J) + \Delta H(J)^{\star}] = \frac{1}{2}[\Delta H(J)^{+} + \Delta H(J)^{\top}]\,, \tag{4.13a}$$

$$[\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J)]^{\top} = \Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J)\,. \tag{4.13b}$$

Thus from (4.13a), (3.19b) and (4.6), it follows that

$$\begin{aligned} &\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J) \\ &\quad= \frac{J^2}{8l}\big[\big(\Delta H(J)^{+} + \Delta H(J)^{\top}\big)\Omega(A(J)^{\star} - A(J)) \\ &\qquad + (A(J)^{\star} - A(J))\big(\Delta H(J)^{+} + \Delta H(J)^{\top}\big)\Omega\big]\Omega \\ &\quad= \frac{J^2}{4l}[\Delta H(J)^{+}\Omega A(J)^{\star} - \Delta H(J)^{\top}\Omega A(J) \\ &\qquad + A(J)^{\star}\Delta H(J)^{\top}\Omega - A(J)\Delta H(J)^{+}\Omega]\Omega \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma}{4l}\bigg[\Omega\frac{\partial}{\partial l}F(l;J)^{+-1}\Omega F(l;J)^{\star-1}\Omega + \Omega\frac{\partial}{\partial l}F(l;J)^{\top-1}\Omega F(l;J)^{-1}\Omega \\
&\quad - \frac{1}{\lambda(l)^2}\bigg(A(J)^\star\Omega\frac{\partial}{\partial l}F(l;J)^{\top-1}\Omega F(l;J)^{-1}A(J)^\star\Omega \\
&\quad + A(J)\Omega\frac{\partial}{\partial l}F(l;J)^{+-1}\Omega F(l;J)^{\star-1}A(J)\Omega\bigg)\bigg],
\end{aligned}
$$

then from (4.13b), (3.19b) and (4.6) we obtain

$$
\begin{aligned}
&\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J) \\
&\quad = \frac{1}{2}\big[\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J) \\
&\qquad + (\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J))^\top\big] \\
&\quad = \frac{\sigma J^2}{8l}\bigg[\bigg(\frac{\partial}{\partial l}A(J) + \frac{\partial}{\partial l}A(J)^\star\bigg) \\
&\qquad - \frac{1}{\lambda(l)^2}\bigg(A(J)^\star\frac{\partial}{\partial l}A(J)A(J)^\star + A(J)\frac{\partial}{\partial l}A(J)^\star A(J)\bigg)\bigg]\Omega \\
&\quad = \frac{\sigma J^2}{8l\lambda(l)^2}\bigg[2\lambda(l)^2\frac{\partial}{\partial l}\big(A(J)+A(J)^\star\big) - \frac{\partial}{\partial l}(\lambda(l)^2)\big(A(J)+A(J)^\star\big)\bigg]\Omega \\
&\quad = -\frac{2\sigma}{\lambda(l)^2}J^2\rho^2\Omega\,.
\end{aligned}
\tag{4.14}
$$

This result shows that the double transformation (4.10) preserves the condition (3.1b) provided $\Delta\rho = \frac{\sigma}{\lambda(l)^2}\rho$, and we can also verify, by direct calculations, that ${}^*d(\Delta\rho) = d(\Delta z)$ as desired.

Now we consider the equation satisfied by the transformed fields. From (3.10), (3.13), (3.14), (4.12) and (4.14), we have

$$
\begin{aligned}
2(\Delta z + \Delta\rho^*)dH(J) &= 2\sigma(z+\rho^*)\Lambda(l)^{-1}dH(J) \\
&= \frac{\sigma}{l}(H(J)+H(J)^\top)\Omega\Gamma(l;J)\,.
\end{aligned}
\tag{4.15}
$$

Moreover, from (4.10), (3.13), (3.14) and (3.17) we obtain

$$
d\Delta H(J) = \sigma\dot{\Gamma}(l;J) + \sigma J^2[\Gamma(l;J)\Omega,\ \dot{F}(l;J)F(l;J)^{-1}]\Omega\,. \tag{4.16}
$$

Multiplying (4.16) from left by $2(z+\rho^*)$ and using (3.10) and (4.16) again, it follows that

$$
\begin{aligned}
2(z+\rho^*)d\Delta H(J) &= \sigma[(H(J)+H(J)^\top)\Omega, \dot{F}(l;J)F(l;J)^{-1}]\Gamma(l;J) \\
&\quad + (H(J)+H(J)^\top)\Omega\, d\Delta H(J)\,.
\end{aligned}
\tag{4.17}
$$

On the other hand, from (4.10), (3.12), (3.16) and (3.19b) we have

$$
\begin{aligned}
&(\Delta H(J)+\Delta H(J)^{\top})\Omega\, dH(J)\\
&\quad = J^2\sigma l^{-1}[\dot F(l;J)F(l;J)^{-1}\Omega+\Omega F(l;J)^{\top -1}\dot F(l;J)^{\top}]\Omega A(J)\Gamma(l;J)\\
&\quad = \sigma[(H(J)+H(J)^{\top})\Omega, \dot F(l;J)F(l;J)^{-1}]\Gamma(l;J)\\
&\qquad + \sigma l^{-1}(H(J)+H(J)^{\top})\Omega\Gamma(l;J)\,.
\end{aligned}
\tag{4.18}
$$

Finally, (4.15), (4.17) and (4.18) give

$$
\begin{aligned}
&2(\Delta z+\Delta\rho^*)dH(J)+2(z+\rho^*)d\Delta H(J)\\
&\quad = (\Delta H(J)+\Delta H(J)^{\top})\Omega\, dH(J)+(H(J)+H(J)^{\top})\Omega\, d\Delta H(J)\,.
\end{aligned}
\tag{4.19}
$$

The above results show that (4.10) is indeed a double symmetry transformation of Eq. (3.10) with conditions (3.1a), (3.1b).

Similarly, we can prove that (4.11), which gives $\tilde\Delta z=\frac{\epsilon s}{\tilde\lambda(s)^2}[z(s-2z)-2\rho^2]$ and $\tilde\Delta\rho=\frac{\epsilon s^2}{\tilde\lambda(s)^2}\rho$, is also a double symmetry transformation of the EKR theory.

5. Infinite-Dimensional Algebra Structures of the Double Symmetries

From the structures of the double transformations (4.2) and (4.9), we expand the right-hand sides of them in powers of l and s, respectively, as

$$\delta H(J)=\sum_{n=0}^{\infty} l^n\delta^{(n)}H(J)\,, \tag{5.1a}$$

$$\tilde\delta H(J)=\sum_{m=1}^{\infty} s^m\tilde\delta^{(m)}H(J)\,, \tag{5.1b}$$

where the analytic property of $F(l;J)$, $\tilde F(s;J)$ around $l=0$, $s=0$ is noted. Each of $\delta^{(n)}$ and $\tilde\delta^{(m)}$ satisfies the same equations and conditions as δ and $\tilde\delta$ do, thus we have, in fact, constructed infinite many infinitesimal double hidden symmetry transformations of the EKR theory. The algebraic structures of these transformations can be obtained as follows. Noticing the dependence of (4.2), (4.9) on the parameters l, s and the infinitesimal constants α^a in T, we denote the corresponding transformations by $\delta_\alpha(l)$, $\tilde\delta_\alpha(s)$, respectively. Thus we have

$$
\begin{aligned}
&[\delta_\alpha(l),\delta_\beta(l')]H(J)\\
&\quad = -J^2\frac{1}{l}\left[\delta_\beta(l')F(l;J)F(l;J)^{-1}, F(l;J)T_\alpha F(l;J)^{-1}\right]\Omega\\
&\qquad + J^2\frac{1}{l'}\left[\delta_\alpha(l)F(l';J)F(l';J)^{-1}, F(l';J)T_\beta F(l';J)^{-1}\right]\Omega\,,
\end{aligned}
\tag{5.2}
$$

$$
\begin{aligned}
&[\delta_\alpha(l), \tilde{\delta}_\beta(s)]H(J) \\
&\quad = -J^2 \frac{1}{l} [\tilde{\delta}_\beta(s) F(l;J) F(l;J)^{-1}, F(l;J) T_\alpha F(l;J)^{-1}]\Omega \\
&\qquad - J^2 s [\delta_\alpha(l) \tilde{F}(s;J) \tilde{F}(s;J)^{-1}, \tilde{F}(s;J) T_\beta \tilde{F}(s;J)^{-1}]\Omega \,,
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
&[\tilde{\delta}_\alpha(s), \tilde{\delta}_\beta(s')]H(J) \\
&\quad = J^2 s [\tilde{\delta}_\beta(s') \tilde{F}(s;J)\, \tilde{F}(s;J)^{-1}, \tilde{F}(s;J) T_\alpha \tilde{F}(s;J)^{-1}]\Omega \\
&\qquad - J^2 s' [\tilde{\delta}_\alpha(s) \tilde{F}(s';J) \tilde{F}(s';J)^{-1}, \tilde{F}(s';J) T_\beta \tilde{F}(s';J)^{-1}]\Omega \,,
\end{aligned} \tag{5.4}
$$

where $T_\alpha = \alpha^a T_a$, $\delta(l')F(l;J) = F(l, H(J) + \delta(l')H(J); J) - F(l, H(J); J)$, etc.

To obtain the above commutators explicitly, we need the variations of $F(l;J)$, $\tilde{F}(s;J)$ induced by $\delta(l')H(J)$, $\tilde{\delta}(s')H(J)$. It may be verified by tedious but straightforward calculations that we can take

$$
\delta_\alpha(l') F(l;J) = \frac{l}{l-l'} [F(l';J) T_\alpha F(l';J)^{-1} - F(l;J) T_\alpha F(l;J)^{-1}] F(l;J) \,, \tag{5.5}
$$

$$
\tilde{\delta}_\alpha(s) F(l;J) = \frac{ls}{1-ls} [\tilde{F}(s;J) T_\alpha \tilde{F}(s;J)^{-1} - F(l;J) T_\alpha F(l;J)^{-1}] F(l;J) \,, \tag{5.6}
$$

$$
\delta_\alpha(l) \tilde{F}(s;J) = \frac{1}{1-ls} [F(l;J) T_\alpha F(l;J)^{-1} - \tilde{F}(s;J) T_\alpha \tilde{F}(s;J)^{-1}] \tilde{F}(s;J) \,, \tag{5.7}
$$

$$
\tilde{\delta}_\alpha(s') \tilde{F}(s;J) = \frac{s'}{s-s'} [\tilde{F}(s';J) T_\alpha \tilde{F}(s';J)^{-1} - \tilde{F}(s;J) T_\alpha \tilde{F}(s;J)^{-1}] \tilde{F}(s;J) \tag{5.8}
$$

such that $F(l;J) + \delta_\alpha(l')F(l;J)$, $F(l;J) + \tilde{\delta}_\alpha(s)F(l;J)$ satisfy the same Eq. (3.17) and conditions (3.18), (3.19) as $F(l;J)$ does; while $\tilde{F}(s;J) + \delta_\alpha(l)\tilde{F}(s;J)$, $\tilde{F}(s;J) + \tilde{\delta}_\alpha(s')\tilde{F}(s;J)$ satisfy the same Eq. (3.25) and conditions (3.26) as $\tilde{F}(s;J)$ does.

Substituting (5.5)–(5.8) into (5.2)–(5.4), using (4.2), (4.9) again and writing $\delta_\alpha(l)H(J) = \alpha^a \delta_a(l)H(J)$, etc., we obtain

$$
[\delta_\alpha(l), \delta_\beta(l')]H(J) = \frac{\alpha^a \beta^b}{l-l'} C^c_{ab} \big(l\delta_c(l)H(J) - l'\delta_c(l')H(J) \big) \,, \tag{5.9}
$$

$$
[\delta_\alpha(l), \tilde{\delta}_\beta(s)]H(J) = \frac{\alpha^a \beta^b}{1-ls} C^c_{ab} \big(ls\delta_c(l)H(J) + \tilde{\delta}_c(s)H(J) \big) \,, \tag{5.10}
$$

$$
[\tilde{\delta}_\alpha(s), \tilde{\delta}_\beta(s')]H(J) = \frac{\alpha^a \beta^b}{s-s'} C^c_{ab} \big(s'\tilde{\delta}_c(s)H(J) - s\tilde{\delta}_c(s')H(J) \big) \,, \tag{5.11}
$$

where C^c_{ab}'s are structure constants of the Lie algebra $o(d,d)$. Writing (5.1a), (5.1b) in the explicitly α related forms as

$$
\delta_\alpha(l)H(J) = \alpha^a \sum_{n=0}^{\infty} l^n \delta_a^{(n)} H(J) \,, \tag{5.12a}
$$

$$
\tilde{\delta}_\alpha(s)H(J) = \alpha^a \sum_{m=1}^{\infty} s^m \tilde{\delta}_a^{(m)} H(J) \,, \tag{5.12b}
$$

and then expanding both sides of (5.9)–(5.11), we finally obtain

$$\left[\delta_a^{(n)}, \delta_b^{(m)}\right] H(J) = C_{ab}^c \delta_c^{(n+m)} H(J)\,, \quad n, m = 0, \pm 1, \pm 2, \dots\,, \tag{5.13}$$

where $\delta_a^{(-m)} H(J) := \tilde{\delta}_a^{(m)} H(J)$ for $m \geq 1$. Thus, the infinite set of symmetry transformations $\{\delta_a^{(n)},\ n = 0, \pm 1, \pm 2, \dots\}$ constitute a double affine Kac–Moody $\widehat{o(d,d)}$ algebra (without center charge).

Now we consider transformations (4.10), (4.11). They can be expanded as

$$\Delta H(J) = \sigma \sum_{n=0}^{\infty} l^n \Delta^{(n)} H(J)\,, \tag{5.14a}$$

$$\tilde{\Delta} H(J) = \epsilon \sum_{m=1}^{\infty} s^m \tilde{\Delta}^{(m)} H(J)\,. \tag{5.14b}$$

Thus we obtain another infinite set of double symmetry transformations $\{\Delta^{(n)}, \tilde{\Delta}^{(m)},\ n = 0, 1, 2, \dots;\ m = 1, 2, \dots\}$ of the EKR theory. To calculate their commutators, we first denote (4.10), (4.11) by $\Delta_\sigma(l)H(J)$, $\tilde{\Delta}_\epsilon(s)H(J)$, respectively, and then have

$$\begin{aligned}
&[\Delta_\sigma(l), \Delta_{\sigma'}(l')]H(J) \\
&\quad = J^2\sigma \frac{\partial}{\partial l}\big(\Delta_{\sigma'}(l')F(l;J)F(l;J)^{-1}\big)\Omega \\
&\qquad - J^2\sigma' \frac{\partial}{\partial l'}(\Delta_\sigma(l)F(l';J)F(l';J)^{-1})\Omega \\
&\qquad + J^2\sigma[\Delta_{\sigma'}(l')F(l;J)F(l;J)^{-1}, \dot{F}(l;J)F(l;J)^{-1}]\Omega \\
&\qquad - J^2\sigma'[\Delta_\sigma(l)F(l';J)F(l';J)^{-1}, \dot{F}(l';J)F(l';J)^{-1}]\Omega\,,
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
&[\Delta_\sigma(l), \tilde{\Delta}_\epsilon(s)]H(J) \\
&\quad = J^2\sigma \frac{\partial}{\partial l}\big(\tilde{\Delta}_\epsilon(s)F(l;J)F(l;J)^{-1}\big)\Omega \\
&\qquad + J^2\epsilon s^2 \frac{\partial}{\partial s}\big(\Delta_\sigma(l)\tilde{F}(s;J)\tilde{F}(s;J)^{-1}\big)\Omega \\
&\qquad + J^2\sigma[\tilde{\Delta}_\epsilon(s)F(l;J)F(l;J)^{-1}, \dot{F}(l;J)F(l;J)^{-1}]\Omega \\
&\qquad + J^2\epsilon s^2[\Delta_\sigma(l)\tilde{F}(s;J)\tilde{F}(s;J)^{-1}, \dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1}]\Omega\,,
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
&[\tilde{\Delta}_\epsilon(s), \tilde{\Delta}_{\epsilon'}(s')]H(J) \\
&\quad = -J^2\epsilon s^2 \frac{\partial}{\partial s}\big(\tilde{\Delta}_{\epsilon'}(s')\tilde{F}(s;J)\tilde{F}(s;J)^{-1}\big)\Omega \\
&\qquad + J^2\epsilon' {s'}^2 \frac{\partial}{\partial s'}\big(\tilde{\Delta}_\epsilon(s)\tilde{F}(s';J)\tilde{F}(s';J)^{-1}\big)\Omega \\
&\qquad - J^2\epsilon s^2[\tilde{\Delta}_{\epsilon'}(s')\tilde{F}(s;J)\tilde{F}(s;J)^{-1}, \dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1}]\Omega \\
&\qquad + J^2\epsilon' {s'}^2[\tilde{\Delta}_\epsilon(s)\tilde{F}(s';J)\tilde{F}(s';J)^{-1}, \dot{\tilde{F}}(s';J)\tilde{F}(s';J)^{-1}]\Omega\,.
\end{aligned} \tag{5.17}$$

As for $\Delta_\sigma(l')F(l;J)$, $\Delta_\sigma(l)\tilde{F}(s;J)$, etc., we propose

$$\Delta_\sigma(l')F(l;J) = \sigma\frac{l}{l-l'}[l\dot{F}(l;J)F(l;J)^{-1} - l'\dot{F}(l';J)F(l';J)^{-1}]F(l;J)\,, \tag{5.18}$$

$$\tilde{\Delta}_\epsilon(s)F(l;J) = \epsilon\frac{ls}{ls-1}\left[l\dot{F}(l;J)F(l;J)^{-1} + s\dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1} + \frac{1}{2}\right]F(l;J)\,, \tag{5.19}$$

$$\Delta_\sigma(l)\tilde{F}(s;J) = \sigma\frac{1}{ls-1}\left[s\dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1} + l\dot{F}(l;J)F(l;J)^{-1} + \frac{1}{2}\right]\tilde{F}(s;J)\,, \tag{5.20}$$

$$\tilde{\Delta}_\epsilon(s')\tilde{F}(s;J) = \epsilon\frac{s'}{s-s'}[s\dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1} - s'\dot{\tilde{F}}(s';J)\tilde{F}(s';J)^{-1}]\tilde{F}(s;J)\,. \tag{5.21}$$

By some lengthy but straightforward calculations, it can be verified that (5.18), (5.19) are double symmetry transformations of Eq. (3.17) with conditions (3.18), (3.19); while (5.20), (5.21) are double symmetry transformations of Eq. (3.25) with conditions (3.26).

Substituting (5.18)–(5.21) into (5.15)–(5.17) and using (4.10), (4.11) again, it follows that

$$\begin{aligned}
&[\Delta_\sigma(l), \Delta_{\sigma'}(l')]H(J)\\
&\quad = \sigma\frac{\partial}{\partial l}\left[\frac{l}{l-l'}\big(l\Delta_{\sigma'}(l)H(J) - l'\Delta_{\sigma'}(l')H(J)\big)\right]\\
&\qquad - \sigma'\frac{\partial}{\partial l'}\left[\frac{l'}{l'-l}\big(l'\Delta_\sigma(l')H(J) - l\Delta_\sigma(l)H(J)\big)\right],
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
&[\Delta_\sigma(l), \tilde{\Delta}_\epsilon(s)]H(J)\\
&\quad = \sigma\frac{\partial}{\partial l}\left[\frac{ls}{ls-1}\big(l\Delta_\epsilon(l)H(J) - s^{-1}\tilde{\Delta}_\epsilon(s)H(J)\big)\right]\\
&\qquad + \epsilon s^2\frac{\partial}{\partial s}\left[\frac{1}{ls-1}\big(l\Delta_\sigma(l)H(J) - s^{-1}\tilde{\Delta}_\sigma(s)H(J)\big)\right],
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
&[\tilde{\Delta}_\epsilon(s), \tilde{\Delta}_{\epsilon'}(s')]H(J)\\
&\quad = \epsilon s^2\frac{\partial}{\partial s}\left[\frac{s'}{s-s'}\big(s^{-1}\tilde{\Delta}_{\epsilon'}(s)H(J) - s'^{-1}\tilde{\Delta}_{\epsilon'}(s')H(J)\big)\right]\\
&\qquad - \epsilon' s'^2\frac{\partial}{\partial s'}\left[\frac{s}{s'-s}\big(s'^{-1}\tilde{\Delta}_\epsilon(s')H(J) - s^{-1}\tilde{\Delta}_\epsilon(s)H(J)\big)\right].
\end{aligned} \tag{5.24}$$

By using (5.14a), (5.14b) to expand both sides of (5.22)–(5.24), we obtain

$$[\Delta^{(m)}, \Delta^{(n)}]H(J) = (m-n)\Delta^{(m+n)}H(J)\,, \quad m, n = 0, \pm1, \pm2, \dots\,, \tag{5.25}$$

where we have written $\Delta^{(-n)}H(J) := \tilde{\Delta}^{(n)}H(J)$ for $n \geq 1$. This shows that the infinite set of symmetry transformations $\{\Delta^{(n)},\ n = 0,\ \pm1,\ \pm2, \dots\}$ constitute a double Virasoro algebra (without center charge).

Next we investigate the commutators between the members of $\{\delta^{(m)}\}$ and $\{\Delta^{(n)}\}$. For example, from (4.2), (4.10), (5.5) and (5.18) we have, by some calculations

$$[\Delta_\sigma(l), \delta_a(s)]H(J) = \sigma\frac{\partial}{\partial l}\left[\frac{l}{l-s}(l\delta_a(l)H(J) - s\delta_a(s)H(J))\right] - \sigma\frac{l}{l-s}\frac{\partial}{\partial l}(l\delta_a(l)H(J)) + \sigma\frac{s}{l-s}\frac{\partial}{\partial s}(s\delta_a(s)H(J))\,. \quad (5.26)$$

Similarly, we can give out the expressions of $[\Delta_\sigma(l), \tilde{\delta}_a(s)]H(J)$, $[\tilde{\Delta}_\sigma(l), \delta_a(s)]H(J)$ and $[\tilde{\Delta}_\sigma(l), \tilde{\delta}_a(s)]H(J)$. Then by using (5.12a), (5.12b) and (5.14a), (5.14b) to expand both sides of these results, we finally obtain

$$[\Delta^{(m)},\ \delta_a^{(n)}]H(J) = -n\delta_a^{(m+n)}H(J)\,, \quad m, n = 0, \pm 1, \pm 2, \ldots\,. \quad (5.27)$$

Equations (5.13), (5.25) and (5.27) show that the symmetry transformations (4.2), (4.9)–(4.11) give a double representation of semidirect product of the affine $\widehat{o(d,d)}$ and Virasoro algebras. These give expression to that the infinite-dimensional symmetry structures of the EKR theory contain not only the double Kac–Moody $\widehat{o(d,d)}$ algebra but also the double Virasoro algebra and demonstrate that the theory under consideration possesses richer symmetry structures than previously expected.

6. Summary and Discussions

By using the so-called ED-complex function method,[32] the previously found doubleness symmetry[26] of the dimensionally reduced EKR theory is further exploited in the present paper. A double-complex H-potential $H(J)$ is introduced in (3.8) and the motion equations of the EKR theory are written as a double-complex form (3.10). Moreover, we also find that the theory under consideration has more double symmetries which make us be able to establish a pair of ED-complex HE-type linear systems (3.17)–(3.19), (3.25), (3.26). Based on these linear systems, we explicitly construct the set of double symmetry transformations (4.2), (4.9)–(4.11). These symmetries are verified to constitute double infinite-dimensional Lie algebras, each of which is a semidirect product of the Kac–Moody $\widehat{o(d,d)}$ and Virasoro algebras. These results show that the ED-complex method is necessary and more effective. Some of the results in this paper cannot be obtained by the usual (non-ED-complex) scheme.

We would like to indicate that although Eqs. (3.17) and (3.25) are, in form, interrelated by $t \leftrightarrow w = 1/t$, the analytic properties of $F(t; J)$ and $\tilde{F}(w; J)$ as well as the conditions (3.18), (3.19) and (3.26) do not have this interrelation, therefore as whole ED-complex linear systems they are different and give rise to different parts of symmetries of the EKR theory. From (5.12)–(5.14), (5.25) and (5.27), we can see that the double symmetry transformations (4.2), (4.10) constructed by using $F(t; J)$ give only the "nonnegative index" part of the "complete" double infinite-dimensional Kac–Moody–Virasoro symmetry algebra and constitute a double subalgebra, while the double symmetry transformations (4.9), (4.11) constructed by using $\tilde{F}(w; J)$ give the "negative index" part of the "complete" double infinite-dimensional symmetry algebra and do not constitute any subalgebra (only is a double subset of the whole symmetry algebra).

Finite symmetry transformations relating to the above infinitesimal ones and soliton solutions of the studied theory need more and further investigations and will be considered in some forthcoming works.

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